

PHYS 3322 Modern Laboratory Methods I

Transfer Functions of AC R, RC, & RL Circuits

Purpose

The study of AC circuits entails an understanding of reactive impedance as well as the effect that each circuit element has on the overall impedance of the circuit. The impedance of reactive elements, such as capacitors and inductors, is frequency dependent and allows the construction of passive filters for signal attenuation. This theory section develops the transfer functions that relate the voltage across circuit elements to the applied or driving sinusoidal voltage in R, RC, and RL AC circuits. The transfer functions developed in this section will be used as a comparison to the experimentally determined transfer functions of Experiment 3.

Background

In an AC circuit, a sinusoidal driving voltage having amplitude V_o may be expressed as

$$V(t) = V_o \cos \omega t = \text{Re}[V_o e^{i\omega t}] \quad , \quad (1)$$

where the complex identity $\text{Re}[e^{\pm i\theta}] = \text{Re}[\cos \theta \pm i \sin \theta] = \cos \theta$ is used. In this expression, ω is the angular frequency and $\omega = 2\pi f$, where f is the frequency in units of Hertz. In a circuit containing reactive components, the applied voltage and current are generally not in phase. With $V(t)$ expressed as in equation 1, the current $I(t)$ is also sinusoidal having an amplitude I_o that depends not only on the magnitude $V(t)$ but also on the total impedance Z of the circuit. Hence,

$$I(t) = \frac{V_o}{Z} \cos(\omega t + \phi) = I_o \cos(\omega t + \phi) = \text{Re}[I_o e^{i(\omega t + \phi)}] \quad (2)$$

In this expression, ϕ is the phase between the applied voltage and the current in the circuit.

When performing the analysis of electrical circuits in order to determine the phase and total impedance, the technique is the same if the trigonometric (real) expressions are used or if the complex expressions are used. When using the complex expressions, all that must be done is to take the real part in the end. For instance, given a circuit consisting of an AC voltage source in series with an ideal resistor R (Figure 1a) the current can be determined using Kirchhoff's loop rule and Ohm's Law. Hence, the relationship, where $V_{IN} = V(t)$ is the AC voltage,

$$V_{IN} = IR \quad (3)$$

Using the real and complex forms, the input voltage is expressed as

	<i>Real form</i>	<i>Complex form</i>
	$V_{IN} = V_o \cos(\omega t)$	$V_{IN} = V_o e^{i\omega t} \quad (4)$
Solving for the current yields,	$I = \frac{V_o}{R} \cos(\omega t) = I_o \cos(\omega t)$	$I = \frac{V_o}{R} e^{i\omega t} = I_o e^{i\omega t} \quad (5)$
Taking the real part of these expressions	$I = I_o \cos(\omega t)$	$I = \text{Re}[I_o e^{i\omega t}] = I_o \cos(\omega t) \quad (6)$

In both cases, the end result is the same and it can be seen that the current is sinusoidal having amplitude V_o/R and angular frequency ω and also that the current is in phase with the applied voltage V_{IN} .

Transfer Functions of AC R, RC, & RL Circuits

The situation is a little more involved mathematically if we were to analyze an AC voltage source in series with a resistor and a capacitor, a series RC circuit (Figure 1b). For this circuit, Kirchhoff's loop rule and Ohm's Law yields

$$V_{IN} = IR + IZ_C \quad . \quad (7)$$

Z_C is the impedance of the capacitor and itself is a function of frequency,

$$Z_C = \frac{1}{i\omega C} \quad . \quad (8)$$

Solving (7) for the current yields,

$$I = \frac{V_{IN}}{R + Z_C} = V_{IN} \frac{1}{R - \frac{i}{\omega C}} = V_{IN} \frac{e^{-i \tan^{-1}\left(-\frac{1}{\omega RC}\right)}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} = V_{IN} \frac{e^{i\left[\frac{\pi}{2} - \tan^{-1}(\omega RC)\right]}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \quad . (8)$$

[The denominator is of the form $a + ib = re^{i\phi}$, where $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}[b/a]$.] In full form equation (8) is

$$I(t, \omega) = \frac{V_o}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} e^{i\left[\omega t + \frac{\pi}{2} - \tan^{-1}(\omega RC)\right]} \quad (9)$$

The current is oscillatory in time and phase shifted by $\frac{\pi}{2} - \tan^{-1}(\omega RC)$ with respect to the driving sinusoidal voltage and has an amplitude given by the pre-exponential factor. The denominator of the pre-exponential factor is the total, but frequency dependent, impedance of this RC circuit. Note that the current, as well as the driving voltage, can be considered to be not only a function of time but also of frequency.

The sinusoidal current is difficult to measure directly but can be obtained indirectly by measuring the voltage across the resistor and using Ohm's Law. To see this, the voltage across the resistor is

$$V_R = IR = V_{IN} \frac{e^{i\left[\frac{\pi}{2} - \tan^{-1}(\omega RC)\right]}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} R \quad . \quad (9)$$

Measuring the voltage across the resistor at some time t and dividing by the resistance yields an instantaneous a measurement of the current.

Dividing V_R by the input or driving voltage V_{IN} , equation (9) yields the expression

$$\frac{V_R}{V_{IN}} = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} e^{i\left[\frac{\pi}{2} - \tan^{-1}(\omega RC)\right]} \quad . \quad (10)$$

This ratio represents the transfer function of V_R ; it is the function that when multiplied by V_{IN} yields V_R for any series RC AC circuit. The transfer function contains the phase shift with respect to V_{IN} and an amplitude factor that depends of the impedance values of the circuit elements and the input frequency.

As it is written, the transfer function yields V_R at all times given V_{IN} . Certainly an oscilloscope can display the complete signal but the easiest way to compare the input voltage to the voltage

drop across the resistor is look at the amplitudes of each and the phase difference between the two. Multiplying the transfer function by its complex conjugate and taking the square root yields the magnitude of the transfer function. Hence,

$$\left| \frac{V_R}{V_{IN}} \right| = \sqrt{\left(\frac{V_R}{V_{IN}} \right) \left(\frac{V_R}{V_{IN}} \right)^*} = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C} \right)^2}} \quad (11)$$

Using an oscilloscope to measure the amplitude of V_R , or for that matter the amplitude of any voltage signal across a circuit element, as well as the amplitude of V_{IN} will determine the experimental values for the magnitude of the transfer function at any frequency. An oscilloscope can also be used to measure the phase shift between V_{IN} and V_R .

Procedure

The write up for this lab consists of determining the current, phase, impedance, and transfer functions for R, RC, and RL circuits **in detail**.

THE R CIRCUIT

For the circuit shown in Figure 1 obtain the following theoretical expressions:

- The magnitude of the current.
- The phase difference between the applied voltage and the current.
- From (a), the total impedance of the circuit.
- The transfer function for resistor.
- Without performing any formal graphing analysis, do a qualitative (rough) sketch of the impedance.

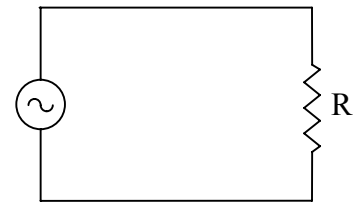


Figure 1. The R circuit

THE RC CIRCUIT

For the circuit shown in Figure 2 obtain the following theoretical expressions:

- The magnitude of the current.
- The phase difference between the applied voltage and the current.
- From (a), the magnitude of the total impedance of the circuit.
- The transfer function for resistor.
- The transfer function for the capacitor.
- Without performing any formal graphing analysis, do a qualitative (rough) sketch of the impedance

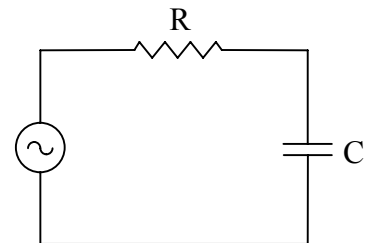


Figure 2. The RC circuit.

THE RL CIRCUIT (CASE I)

For Case I the impedance for the inductor is given by

$$Z_L = i\omega L$$

For the circuit shown in Figure 3 obtain the following theoretical expressions:

- The magnitude of the current.
- The phase difference between the applied voltage and the current.
- From (a), the magnitude of the total impedance of the circuit.
- The transfer function for resistor.
- The transfer function for the inductor.
- Without performing any formal graphing analysis, do a qualitative (rough) sketch of the impedance.

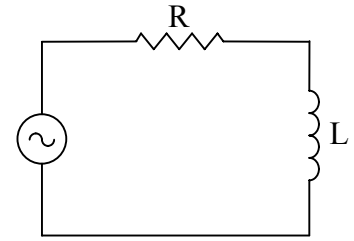


Figure 3. The RL circuit.

THE RL CIRCUIT (CASE II)

For Case II the impedance for the inductor is given by

$$Z_L = R_L + i\omega L$$

Here R_L is the resistance of the inductor. The inclusion of the resistance of the inductor R_L is due to the typical construction of inductors. The inductor is typically made up of a long metallic wire with a small cross-section that has been coiled. As such the resistance can be significant.

- Does the expression for Z_L given above behave similar to that of an ideal inductor when taking into account the frequency behavior, say, from zero frequency (DC) to very large frequencies?
- How does the inductance L vary with the number of turns in the coil that make up the inductor? What should happen if we were to cut an inductor in half? Does the expression for Z_L satisfy this?

For the circuit shown in Figure 4 obtain the following theoretical expressions:

- The magnitude of the current.
- The phase difference between the applied voltage and the current.
- From (a), the magnitude of the total impedance of the circuit.
- The transfer function for resistor.
- The transfer function for the inductor.
- Without performing any formal graphing analysis, do a qualitative (rough) sketch of the impedance.

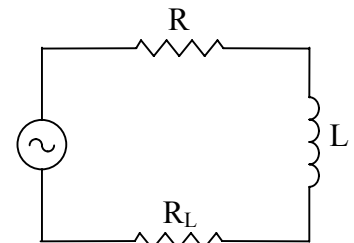


Figure 4. The RL circuit.